An Analytical, Practical and Application Approach about Perturbation Theory

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ABSTRACT

Present Paper on An Analytical, Practical and Application approach about Perturbation theory explains the need and also reflects the application of perturbation theory beside it also exhibit how important the utility of Perturbation theory, which is a general mathematical method that are used to find an approximate solution to a problem which cannot be solved exactly, by starting from the exact solution of a related problem. Perturbation theory is applicable if the problem at hand can be formulated by adding a "small" term to the mathematical description of the exactly solvable problem.

Perturbation theory leads to an expression for the desired solution in terms of a f*ormal power series* in some "small" parameter – known as a perturbation series – that quantifies the deviation from the exactly solvable problem. The leading term in this power series is the solution of the exactly solvable problem, while further terms describe the deviation in the solution, due to the deviation from the initial problem. Formally, we have for the approximation to the full solution A, a series in the small parameter (here called ϵ), like the following:

$$A = A_0 + \epsilon^1 A_1 + \epsilon^2 A_2 + \cdots$$

In this example, A_0 would be the known solution to the exactly solvable initial problem and A_1, A_2,... represent the higher-order terms which may be found iteratively by some systematic procedure. For small ϵ these higher-order terms in the series become successively smaller. An approximate "perturbation solution" is obtained by truncating the series, usually by keeping only the first two terms, the initial solution and the "first-order" perturbation correction:

$$A \approx A_0 + \epsilon A_1$$

Keywords
Perturbation theory, Keplerian ellipse, numerical analysis, celestial mechanics, older civilizations

1. INTRODUCTION

Perturbation theory is closely related to methods used in numerical analysis. The earliest use of what would now be called *perturbation theory* was to deal with the otherwise unsolvable mathematical problems of celestial mechanics: Newton's solution for the orbit of the Moon, which moves noticeably differently from a simple Keplerian ellipse because of the competing gravitation of the Earth and the Sun.

Perturbation methods start with a simplified form of the original problem, which is *simple enough* to be solved exactly. In celestial mechanics, this is usually a Keplerian ellipse. Under non relativistic gravity, an ellipse is exactly correct when there are only two gravitating bodies (say, the Earth and the Moon) but not quite correct when there are three or more objects (say, the Earth, Moon, Sun, and the rest of the solar system).

The solved, but simplified problem is then *"perturbed"* to make the conditions that the perturbed solution actually satisfies closer to the real problem, such as including the gravitational attraction of a third body (the Sun). The "conditions" are a formula (or several) that represent reality, often something arising from a physical law like Newton's second law, the force-acceleration equation:

$$\mathbf{F} = m\mathbf{a}$$

In the case of the example, the force $\mathbf{F}$ is calculated based on the number of gravitationally relevant bodies; the acceleration $\mathbf{a}$ is obtained, using calculus, from the path of the Moon in its orbit. Both of these come in two forms: approximate values for force and acceleration, which result from simplifications, and hypothetical exact values for force and acceleration, which would require the complete answer to calculate.

The slight changes that result from accommodating the perturbation, which themselves may have been simplified yet again, are used as corrections to the approximate solution. Because of simplifications introduced along every step of the way, the corrections are never perfect,

and the conditions met by the corrected solution do not perfectly match the equation demanded by reality, but even one cycle of corrections often provides a remarkably better approximate answer to what the real solution should be.

There is no requirement to stop at only one cycle of corrections. A partially corrected solution can be re-used as the new starting point for yet another cycle of perturbations and corrections. In principle, cycles of finding increasingly better corrections could go on indefinitely. In practice, one typically stops at one or two cycles of corrections. The usual difficulty with the method is that the corrections progressively make the new solutions very much more complicated, so each cycle is much more difficult to manage than the previous cycle of corrections. Isaac Newton is reported to have said, regarding the problem of the Moon's orbit, that *"It caused my head to ache."*

This general procedure is a widely used mathematical tool in advanced sciences and engineering: start with a simplified problem and gradually add corrections that make the formula that the corrected problem matches closer and closer to the formula that represents reality. It is the natural extension to mathematical functions of the "guess, check, and fix" method used by older civilizations to compute certain numbers, such as square root

2.0 OBJECTIVE OF STUDY:

1.To Study the "mathematical description" are: an algebraic equation, a differential equation (e.g., the equations of motion in celestial mechanics or a wave equation), a free energy (in statistical mechanics), a Hamiltonian operator (in quantum mechanics).

2. To Study the kind of solution to be found perturbatively: the solution of the equation (e.g., the trajectory of a particle), the statistical average of some physical quantity (e.g., average magnetization), the ground state energy of a quantum mechanical problem.

3. To Study the exactly solvable problems to start with: linear equations, including linear equations of motion (harmonic oscillator, linear wave equation), statistical or quantum-mechanical systems of non-interacting particles (or in general, Hamiltonians or free energies containing only terms quadratic in all degrees of freedom).

3.0 LITERATURE STUDY AND REVIEW

Perturbation theory has its roots in early celestial mechanics, where the theory of epicycles was used to make small corrections to the predicted paths of planets. Curiously, it was the need for more and more epicycles that eventually led to the 16th century Copernican revolution in the understanding of planetary orbits. The development of basic perturbation theory for differential equations was fairly complete by the middle of the 19th century. It was at that time that Charles-Eugène Delaunay was studying the perturbative expansion for the Earth-Moon-Sun system, and discovered the so-called "problem of small denominators". Here, the denominator appearing in the *n*'th term of the perturbative expansion could become arbitrarily small, causing the *n*'th correction to be as large as or larger than the first-order correction. At the turn of the 20th century, this problem led Henri Poincare to make one of the first deductions of the existence of chaos, or what is prosaically called the "butterfly effect": that even a very small perturbation can have a very large effect on a system.

Perturbation theory saw a particularly dramatic expansion and evolution with the arrival of quantum mechanics. Although perturbation theory was used in the semi-classical theory of the Bohr atom, the calculations were monstrously complicated, and subject to somewhat ambiguous interpretation. The discovery of Heisenberg's matrix mechanics allowed a vast simplification of the application of perturbation theory. Notable examples are the Stark effect and the Zeeman Effect, which have a simple enough theory to be included in standard undergraduate textbooks in quantum mechanics. Other early applications include the fine structure and the hyperfine structure in the hydrogen atom.

In modern times, perturbation theory underlies much of quantum chemistry and quantum field theory. In chemistry, perturbation theory was used to obtain the first solutions for the helium atom. In the middle of the 20th century, Richard Feynman realized that the perturbative expansion could be given a dramatic and beautiful graphical representation in terms of what are now called Feynman diagrams. Although originally applied only in quantum field theory, such diagrams now find increasing use in any area where perturbative expansions are studies

A partial resolution of the small-divisor problem was given by the statement of the KAM theorem in 1954. Developed by Andrey Kolmogorov, Vladimir Arnold and Jürgen Moser, this theorem stated the conditions under which a system of partial differential equations will have only mildly chaotic behavior under small perturbations.

In the late 20th century, broad dissatisfaction with perturbation theory in the quantum physics community, including not only the difficulty of going beyond second order in the expansion, but also questions about whether the perturbative expansion is even convergent, has led to a strong interest in the area of non-perturbative analysis, that is, the study of exactly solvable models. The prototypical model is the Korteweg–de Vries equation, a highly non-linear equation for which the interesting solutions, the solitons, cannot be reached by perturbation theory, even if the perturbations were carried out to infinite order. Much of the theoretical work in non-perturbative analysis goes under the name of quantum groups and non-commutative

4.0 EXTENSIVE STUDY OF PERTURBATION THEORY AND ITS CHARACTERISTICS:

This section deals with the detailed study about perturbation theory, its order, First order non singular Perturbation theory, perturbation theory of degenerate state, and the Example of degenerate perturbation theory - Stark effect in resonant rotating wave.

4.1 Perturbation orders

The standard exposition of perturbation theory is given in terms of the order to which the perturbation is carried out: first-order perturbation theory or second-order perturbation theory, and whether the perturbed states are degenerate (that is, singular), in which case extra care must be taken, and the theory is slightly more difficult.

4.2First-order non-singular perturbation theory

This section develops, in simplified terms, the general theory for the perturbative solution to a differential equation to the first order. To keep the exposition simple, a crucial assumption is made: that the solutions to the unperturbed system are not *degenerate*, so that the perturbation series can be inverted. There are ways of dealing with the degenerate (or *singular*) case; these require extra care.

Suppose one wants to solve a differential equation of the form

$$Dg(x) = \lambda g(x)$$

where D is some specific differential operator, and λ is an eigen value. Many problems involving ordinary or partial differential equations can be cast in this form. It is presumed that the differential operator can be written in the form

$$D = D^{(0)} + \epsilon D^{(1)}$$

where ϵ is presumed to be small, and that furthermore, the complete set of solutions for $D^{(0)}$ are known. That is, one has a set of solutions $f_n^{(0)}(x)$, labelled by some arbitrary index n, such that

$$D^{(0)} f_n^{(0)}(x) = \lambda_n^{(0)} f_n^{(0)}(x).$$

Furthermore, one assumes that the set of solutions $\{f_n^{(0)}(x)\}$ form an orthonormal set:

$$\int f_m^{(0)}(x) f_n^{(0)}(x)\, dx = \delta_{mn}$$

with δ_{mn} the Kronecker delta function.

To zeroth order, one expects that the solutions $g(x)$ are then somehow "close" to one of the unperturbed solutions $f_n^{(0)}(x)$. That is,

$$g(x) = f_n^{(0)}(x) + \mathcal{O}(\epsilon)$$

and

$$\lambda = \lambda_n^{(0)} + \mathcal{O}(\epsilon).$$

where $\mathcal{O}$ denotes the relative size, in big-O notation, of the perturbation. To solve this problem, one assumes that the solution $g(x)$ can be written as a linear combination of the $f_n^{(0)}(x)$:

$$g(x) = \sum_m c_m f_m^{(0)}(x)$$

with all of the constants $c_m = \mathcal{O}(\epsilon)$ except for n, where $c_n = \mathcal{O}(1)$. Substituting this last expansion into the differential equation, taking the inner product of the result with $f_n^{(0)}(x)$, and making use of orthogonality, one obtains

$$c_n \lambda_n^{(0)} + \epsilon \sum_m c_m \int f_n^{(0)}(x) D^{(1)} f_m^{(0)}(x)\, dx = \lambda c_n$$

This can be trivially rewritten as a simple linear algebra problem of finding the eigenvalue of a matrix, where

$$\sum_m A_{nm} c_m = \lambda c_n$$

where the matrix elements A_{nm} are given by

$$A_{nm} = \delta_{nm} \lambda_n^{(0)} + \epsilon \int f_n^{(0)}(x) D^{(1)} f_m^{(0)}(x)\, dx$$

Rather than solving this full matrix equation, one notes that, of all the c_m in the linear equation, only one, namely c_n, is not small. Thus, to the first order in ϵ, the linear equation may be solved trivially as

$$\lambda = \lambda_n^{(0)} + \epsilon \int f_n^{(0)}(x) D^{(1)} f_n^{(0)}(x)\, dx$$

since all of the other terms in the linear equation are of order $\mathcal{O}(\epsilon^2)$. The above gives the solution of the eigenvalue to first order in perturbation theory.

The function $g(x)$ to first order is obtained through similar reasoning. Substituting

$$g(x) = f_n^{(0)}(x) + \epsilon f_n^{(1)}(x)$$

so that

$$\left(D^{(0)} + \epsilon D^{(1)}\right)\left(f_n^{(0)}(x) + \epsilon f_n^{(1)}(x)\right) = \left(\lambda_n^{(0)} + \epsilon\lambda_n^{(1)}\right)\left(f_n^{(0)}(x) + \epsilon f_n^{(1)}(x)\right)$$

gives an equation for $f_n^{(1)}(x)$. It may be solved integrating with the partition of unity

$$\delta(x-y) = \sum_n f_n^{(0)}(x) f_n^{(0)}(y)$$

to give

$$f_n^{(1)}(x) = \sum_{m\,(\neq n)} \frac{f_m^{(0)}(x)}{\lambda_n^{(0)} - \lambda_m^{(0)}} \int f_m^{(0)}(y) D^{(1)} f_n^{(0)}(y)\, dy$$

which gives the exact solution to the perturbed differential equation to the first order in the perturbation ϵ. Several important observations can be made about the form of this solution. First, the sum over functions with differences of eigenvalues in the denominator resembles the resolvent in Fredholm theory. This is no accident; the resolvent acts essentially as a kind of Green's function or propagator, passing the perturbation along. Higher-order perturbations resemble this form, with an additional sum over a resolvent appearing at each order.

The form of this solution is sufficient to illustrate the idea behind the small-divisor problem. If, for whatever reason, two eigenvalues are close so that difference $\lambda_n^{(0)} - \lambda_m^{(0)}$ become small, the corresponding term in the sum will become disproportionately large. In particular, if this happens in higher-order terms, the high-order perturbation may become as large or larger in magnitude than the first-order perturbation. Such a situation calls into question the validity of doing a perturbation to begin with. This can be understood to be a fairly catastrophic situation; it is frequently encountered in chaotic dynamical systems, and requires the development of techniques other than perturbation theory to solve the problem.

Curiously, the situation is not at all bad if two or more eigen values are exactly equal. This case is referred to as singular or degenerate perturbation theory. The degeneracy of Eigen values indicates that the unperturbed system has some sort of symmetry, and that the generators of the symmetry commute with the unperturbed differential operator. Typically, the perturbing term does not possess the symmetry; one says the perturbation lifts or breaks the degeneracy. In this case, the perturbation can still be performed; however, one must be careful to work in a basis for the unperturbed states so that these map one-to-one to the perturbed states, rather than being a mixture.

4.3Perturbation theory of degenerate states

One may notice that the problem occurs in the first order perturbation theory when two or more eigen functions of the unperturbed system correspond to one eigenvalue i.e. when the eigenvalue equation becomes

$$D^{(0)} f_{n,i}^{(0)}(x) = \lambda_n^{(0)} f_{n,i}^{(0)}(x).$$

and the index i labels many states with the same eigenvalue $\lambda_n^{(0)}$. Expression for the eigenfunctions having the energy differences in the denominators becomes infinite. In that case the degenerate perturbation theory must be applied. The degeneracy must be removed first for higher order perturbation theory. The function is first assumed to be the linear combination of eigenfunctions with the same eigenvalue only

$$g(x) = \sum_k c_{n,k} f_{n,k}^{(0)}(x)$$

which again from the orthogonality of $f_{n,k}^{(0)}$ leads to the following equation

$$c_{n,i}\lambda_{n,i}^{(0)} + \epsilon \sum_k c_{n,k} \int f_{n,i}^{(0)}(x) D^{(1)} f_{n,k}^{(0)}(x)\, dx = \lambda c_{n,i}$$

for each n. As for the majority of low quantum numbers n the i changes over small range of integers the later equation can be usually solved analytically as at most 4x4 matrix equation. Once the degeneracy is removed the first and any order of the perturbation theory may be further used with respect to the new functions.

4.4 Example of second-order singular perturbation theory

Consider the following equation for the unknown variable x:

$$x = 1 + \epsilon x^5.$$

For the initial problem with $\epsilon = 0$, the solution is $x_0 = 1$. For small ϵ the lowest-order approximation may be found by inserting the ansatz

$$x = x_0 + \epsilon x_1 (+\cdots)$$

Into the equation and demanding the equation to be fulfilled up to terms that involve powers of ϵ higher than the first. This yields $x_1 = 1$. In the same way, the higher orders may be found. However, even in this simple example it may be observed that for (arbitrarily)

small $\epsilon > 0$ there are four other solutions to the equation (with very large magnitude). The reason we don't find these solutions in the above perturbation method is because these solutions diverge when $\epsilon \to 0$ while the ansatz assumes regular behavior in this limit.

The four additional solutions can be found using the methods of singular perturbation theory. In this case this works as follows. Since the four solutions diverge at $\epsilon = 0$, it makes sense to rescale x. We put
$x = y\epsilon^{-\nu}$
such that in terms of y the solutions stay finite. This means that we need to choose the exponent ν to match the rate at which the solutions diverge. In terms of y the equation reads:
$\epsilon^{-\nu} y = 1 + \epsilon^{1-5\nu} y^5$
The 'right' value for ν is obtained when the exponent of ϵ in the prefactor of the term proportional to y is equal to the exponent of ϵ in the prefactor of the term proportional to y^5, i.e. when $\nu = 1/4$. This is called 'significant degeneration'. If we choose ν larger, then the four solutions will collapse to zero in terms of y and they will become degenerate with the solution we found above. If we choose ν smaller, then the four solutions will still diverge to infinity.
Putting $\nu = 1/4$ in the above equation yields:
$y = \epsilon^{1/4} + y^5$
This equation can be solved using ordinary perturbation theory in the same way as regular expansion for x was obtained. Since the expansion parameter is now $\epsilon^{1/4}$ we put:
$y = y_0 + \epsilon^{1/4} y_1 + \epsilon^{1/2} y_2 \cdots$
There are 5 solutions for y_0: 0, 1, -1, i and -i. We must disregard the solution $y = 0$. The case $y = 0$ corresponds to the original regular solution which appears to be at zero for $\epsilon = 0$, because in the limit $\epsilon \to 0$ we are rescaling by an infinite amount. The next term is $y_1 = -1/4$. In terms of x the four solutions are thus given as:
$x = \epsilon^{-1/4} \left[y_0 - 1/4\epsilon^{1/4} + \cdots \right]$

4.5 Example of degenerate perturbation theory - Stark effect in resonant rotating wave

Let us consider the atom of Hydrogen in the electric field rotating with a constant angular frequency ω and the Hamilton operator
$H = H_0 + \epsilon x$
where the unperturbed Hamiltonian is
$H_0 = \mathbf{p}^2/2 - 1/r - \omega L_z,$
and L_z is the operator for the z-component of angular momentum: $L_z = i\hbar \frac{\partial}{\partial \phi}$. The perturbation ϵx can be seen as electric field strength multiplied by one of the space coordinates (This calculation is in atomic units, so that every quantity is dimensionless, indeed $\hbar = 1$).

The eigenvalues of H_0 are
$E_{n,m} = -1/2n^2 - m\omega$
For the lowest eigenstates of Hydrogen $|n, l, m\rangle$, $|1, 0, 0\rangle$ and $|2, 1, 1\rangle$ in the resonance $E_{2,1} - E_{1,0} = 0$ they are therefore both equal $E_{1,0} = -1/2$, while the eigenstates are different. The eigenvalue equation takes the form

$$\begin{bmatrix} E_{1,0} & \epsilon d \\ \epsilon d & E_{1,0} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = E \begin{bmatrix} a \\ b \end{bmatrix}$$

where
$d = \frac{128}{243} a_0$
which leads to the quadratic equation which can be readily solved
$(E_{1,0} - E)^2 - d^2\epsilon^2 = 0$
with the solution
$|\chi 1\rangle = (|1,0,0\rangle + |2,1,1\rangle)/\sqrt{2}$
$E(1) = E_{1,0} + d\epsilon$
$|\chi 2\rangle = (|1,0,0\rangle - |2,1,1\rangle)/\sqrt{2}$
$E(2) = E_{1,0} - d\epsilon$
Those are Stark states in the rotating frame so-called Trojan (higher eigenvalue) and anti-Trojan wavepackets.

5.0 ANALYSIS AND RESULTS SUMMARY:

Both regular and singular perturbation theory are frequently used in physics and engineering. Regular perturbation theory may only be used to find those solutions of a problem that evolve smoothly out of the initial solution when changing the parameter (that are adiabatically connected" to the initial solution).

Perturbation techniques can be also used to find approximate solutions to non-linear differential equations. Examples of techniques used to find approximate solutions to these types of problems are the Lindstedt–Poincaré technique, the method of harmonic balancing, and the method of multiple time scales.

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There is absolutely no guarantee that perturbative methods result in a convergent solution. In fact, asymptotic series are the norm.

6.0 LIMITATIONS OF PERTURBATION THEORY:

1. Perturbation theory can fail when the system can transition to a different "phase" of matter, with a qualitatively different behavior, that cannot be modeled by the physical formulas put into the perturbation theory (e.g., a solid crystal melting into a liquid). In some cases, this failure manifests itself by divergent behavior of the perturbation series. Such divergent series can sometimes be resumed using techniques such as Borel resummation.
2. A well-known example from physics where regular perturbation theory fails is in fluid dynamics when one treats the viscosity as a small parameter. Close to a boundary, the fluid velocity goes to zero, even for very small viscosity (the no-slip condition). For zero viscosity, it is not possible to impose this boundary condition and a regular perturbative expansion amounts to an expansion about an unrealistic physical solution. Singular perturbation theory can, however, be applied here and this amounts to 'zooming in' at the boundaries (using the method of matched asymptotic expansions).

7.0 FUTURE STUDY AND ALLIED APPLICATIONS OF THE PERTURBATION THEORY:

We may enlist the future research study and also the specific utility of the perturbation theory as under :
Cosmological perturbation theory, Dynamic nuclear polarization, Eigen value perturbation, Interval FEM, Orders of approximation , Structural stability, Application in the quantum Chemistry.

Beside the above, our focus of the research and study will be perturbation theory for solving the Non linear differential equations.

8.0 REFERENCES AND BOOKS.

1. Cropper, William H. (2004), *Great Physicists: The Life and Times of Leading Physicists from Galileo to Hawking*, Oxford University Press, p. 34, ISBN 978-0-19-517324-6.

2. Introduction to regular perturbation theory by Eric Vanden- Eijnden (PDF) Perturbation Method of Multiple Scales.

3. Perturbation theory- an analysis By; Dr. H.K.Das, PHI,Delhi-2011 Edition.

4. Advance mathematics . Dr.N.K.Sharma, I.I.T.Kanpur 2009 Ist Edition

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